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Non-Hermitian Hamiltonians in Quantum Theory

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Abstract

$\mathcal{PT}$-Symmetry and pseudo-Hermicity provide an alternative approach to the "axiom" of Dirac Hermicity in Quantum Mechanics, that is more general and includes more types of Hamiltonians to be considered, where they would otherwise be dismissed as non-Hermitian. In this work, the preliminaries of the non-Hermitian treatment in Quantum Mechanics is given, in terms of the $\mathcal{PT}$-Symmetric and pseudo-Hermitian framework and the tools developed are extended to Field Theories with non-Hermitian mass terms, where the problem of non-Hermicity is resolved and a consistent description of a $\mathcal{PT}$-Symmetric Scalar Field Theory is shown.

Contents

1 Introduction

Since the birth of Quantum Mechanics, the Hamiltonian that governs a system's time evolution is said to be Dirac Hermitian [1] (ie. $H = H^\dagger$) in order for the resulting probabilities to be positive and preserved in Hilbert Space. It was in 1998 that the "axiom" of Dirac Hermicity was first formally questioned by Bender and Boettcher [2], showing that a non-Hermitian but $\mathcal{PT}$-Symmetric class of Hamiltonians (ie. the Hamiltonian operator commutes with the parity and time-reversal operator $[\hat{H}, \hat{\mathcal{P}}\hat{\mathcal{T}}] = 0$) in the form of $\hat{H} = \hat{p}^2 - (ix)^N$ have real spectra and can be thought of as an analytic continuation from real to complex phase spaces. Later in 2002, in a series of papers Ali Mostafazadeh proved that every non-Hermitian Hamiltonian that produces a real eigenspectrum is in fact pseudo-Hermitian and moreover, every $\mathcal{PT}$-Symmetric Hamiltonian with a real eigenspectrum belongs to the class of pseudo-Hermitian Hamiltonians[3, 4, 5].

The glaring difficulty that was encountered in the beginning was that the standard inner product used in Quantum Mechanics is not always positive when considering a non-Hermitian Hamiltonian. This was solved when a previously unnoticed symmetry of $\mathcal{PT}$-Symmetric Hamiltonians, described by the $\mathcal{C}$ operator [6] (given its name as it resembles the Charge Conjugation symmetry but should not be confused with it) was utilized in order to define a new inner product, that is positive definite and consequently gives positive probabilities. This inner product, called the $\mathcal{CPT}$ inner product, gave more viability to non-Hermitian theories as physical theories.

$\mathcal{PT}$-Symmetry has also various applications in optics [7, 8, 9], as well as the field of photonics [10, 11, 12]. The useful property of the $\mathcal{PT}$-Symmetric potentials is the gain-loss (or source-sink) behaviour that proved particularly beneficial in experiments of this kind.

Although the work of Bender and Boetcher shed light on the overlooked use of complex Hamiltonians in 1998, they had already been the topic of discussion since 1928 [13, 14, 15] in the field of dissipative systems and scattering theory (as pointed out in [16]).

Recently, $\mathcal{PT}$-Symmetry in Quantum Field Theory has seen a rather fast rise in popularity, where countless papers have been published ranging from fundamental aspects like renormalization [17, 18] to developing fermionic theories with axions [19]. In the early days, the complex cubic ($i\phi^3$) model [20, 21] and the so called "wrong sign" $-\phi^4$ model [22, 23] were considered, where these are the potentials fall under the general form first studied by Bender and Boetcher (for $N = 3$ and $N = 4$). Although these theories had been well studied, it proved difficult to produce physical measurements, as the $\mathcal{C}$ operator in Quantum Field Theory is very hard to define because there are infinitely many eigenfunctions and the definition of the operator is the sum of all these eigenfunctions.

A prescription for field theories equipped with a non-Hermitian mass term was given in [24], where the field $\phi(x)$ is redifined and the Euler-Lagrange equations can be then formulated (as the usual definition of the equations of motion requires the Lagrangian to be Hermitian), thus resulting in a "modified" Noether's theorem for conserved currents. This can be done for scalar field theories as well as for fermionic field theories.

Furthermore, $\mathcal{PT}$-Symmetry has made an impact in Gravitational aspects as well. In his work, P.Mannheim [25, 26] claims that the issue of dark matter is resolved when considering a Conformally symmetric theory, that is implied by a $\mathcal{PT}$-Symmetric action. Also, this method may provide a possible solution to the ghost problem in higher derivative gravity [27]. In [28], it shown that the astrophysical data may provide evidence for a $\mathcal{PT}$-Symmetric Hamiltonian of Conformal Gravity.

Moreover, work in non-Hermitian Holography [29, 30] has been recently published, that uses the principles of non-Hermitian Quantum Theories. Also, in the sector of Quantum Gravity, non-Hermicity has been shown to provide a possible link between classical and quantum measurements [31], by defining minimal and maximal lengths of measurement. These publications and many more in advanced topics may provide solutions to fundamental problems with theories of everything and could one day give rise to a non-Hermitian fundamental theory.

1.1 Thesis Outline

In this thesis, the notion of pseudo-Hermicity and $\mathcal{PT}$-Symmetry in Quantum Mechanics (QM) and Quantum Field Theory (QFT) is explored. In the first chapter, the general ideas and mechanisms of $\mathcal{PT}$-Symmetric QM are presented and an explicit example of a 2×2 Hamiltonian system is given. Furthermore, attention is given to the inner product of such theories, namely the so called $\mathcal{CPT}$ inner product [32]. This definition solves one of the biggest problems in non-Hermitian theories, the existence of negative norms in the Hilbert space, thus resulting in a positive definite metric.

In the third chapter, the ideas of $\mathcal{PT}$-Symmetry are generalized to those of pseudo-Hermicity and an equivalence between the two structures is shown, thus proving that the most general treatment of a non-Hermitian Quantum Mechanical theory falls under the umbrella of pseudo-Hermitian QM and the $\mathcal{PT}$-Symmetric Hamiltonians of interests are a subset of pseudo-Hermitian ones.

Then, the ideas developed in the first chapters are used to build a Quantum Field Theory(QFT) in chapter four. Initially, a Lagrangian with a non-Hermitian mass matrix is considered [24] and the implications of non-Hermicity are discussed, in terms of the consistency of the Euler-Lagrange equations for a field theory and the consequent conserved currents. Moreover, an alternative method is discussed in this chapter, that was very recently published [33] and gives a very fundamental way of building a "healthy" Field Theory by constructing a new kind of field that transforms in the 'dual' representation and is invariant under the full action of the proper Poincarè group $ISO(1,3)^{\uparrow}$. This is done by extending the standard Poincarè algebra to that of a non-Hermitian theory, which as proved in [33] results in the algebra having non-Hermitian generators. This new method gives a more solid foundation for the field of pseudo-Hermitian QFTs. Finally, the tools for constructing a scalar field theory are given and a concrete example of a $\mathcal{PT}$-Symmetric Scalar Field Theory is illustrated, that is identical to that of the previous treatment but now has been defined using the 'dual' field.

2 $\mathcal{PT}$-Symmetric Quantum Mechanics

The central idea of this formulation is to weaken the axiom of the Hamiltonian's Hermicity and replace it with the more general (and more physical) argument of $\mathcal{PT}$-Symmetry (space and time reflection invariance), thus achieving new theories that could be useful for giving physical descriptions. Recovering the properties of a Hermitian quantum theory requires these symmetries to be unbroken, as the eigenvalues are real and non-complex[6] and thus, the interpretation of results is the same as in the Hermitian case.

2.1 $\mathcal{P}$ and $\mathcal{T}$ operators

The operators act on position and momentum operators $\hat{x}$ and $\hat{p}$ as

$$\mathcal{P} : \hat{x} \to -\hat{x}, \hat{p} \to -\hat{p} \tag{1}$$

$$\mathcal{T} : \hat{x} \to \hat{x}, \hat{p} \to -\hat{p}, i \to -i \tag{2}$$

noting that the time reversal operator $\mathcal{T}$ also changes the sign of i, so that the usual canonical commutator relations are preserved (see 6.1 Appendix A) and $\mathcal{P}$ is linear whereas $\mathcal{T}$ is anti-linear.

The two main properties of these operators are that they commute with each other, ie. $[\mathcal{P}, \mathcal{T}] = 0$ and that $\mathcal{P}^2 = \mathcal{T}^2 = 0$.

One of the central principles of $\mathcal{PT}$-Symmetric theories is that if the symmetry remains unbroken, then all the eigenvalues of the Hamiltonian are positive. A short proof for this statement can be written out as follows [6]:
Let a Hamiltonian H be $\mathcal{PT}$-Symmetric ($[H, \mathcal{PT}] = 0$) and its energy eigenstate ϕ has eigenvalue E:

$$H\phi = E\phi \tag{3}$$

then as H commutes with $\mathcal{PT}$, ϕ is also an eigenstate with some eigenvalue λ:

$$\mathcal{PT}\phi = \lambda\phi \tag{4}$$

By multiplying equation 4 with $\mathcal{PT}$ and using the property that $\mathcal{P}$ and $\mathcal{T}$ commute and $\mathcal{P}^2 = \mathcal{T}^2 = 1$:

$$\mathcal{PTPT}\phi = \mathcal{PT}\lambda\phi \Rightarrow \phi = |\lambda|^2\phi \tag{5}$$

Thus, the eigenvalue λ can be of the form of a pure phase $\lambda = e^{ia}$ for $a \in \mathbb{R}$ as $\lambda^*\lambda = 1$. Now, by multiplying equation 3 by $\mathcal{PT}$ from the left and as H commutes with $\mathcal{PT}$:

$$\mathcal{PT}H\phi = \mathcal{PT}E\phi \Rightarrow E\phi = E^*\phi \tag{6}$$

Hence, the eigenvalue E is real.

In this short proof, the assumption that $\mathcal{PT}$ symmetry is exact (ie. unbroken), ensures that ϕ is the eigenstate of H and $\mathcal{PT}$, as $\mathcal{PT}$ is an anti-linear operator and diagonalization by the same eigenstates of H is ensured.

2.1.1 An example of real spectra

A good example and the first Hamiltonian that was studied [2] is:

$$H = \hat{p}^2 + ix^3 \tag{7}$$

This is clearly $\mathcal{P}$ and $\mathcal{T}$ symmetric $H = H^{\mathcal{PT}}$ and not Dirac Hermitian $H \neq H^\dagger$. This particular class of Hamiltonians was used earlier [34] in order to provide a viable solution to the Yang–Lee edge singularity and the reality of the eigenspectrum was shown, hence not dismissing the solution.

Following Bender and Boetcher, a δ-expansion can be applied to the Hamiltonian 7:

$$H = \hat{p}^2 + x^2(ix)^\delta \tag{8}$$

where δ is a real parameter and H remains $\mathcal{PT}$-symmetric, as long as the reality condition of δ is satisfied. It is obvious that if the parameter δ is set to 0, then the Hamiltonian is Hermitian and reduces to that of the harmonic oscillator. After a rigorous numerical analysis [2], it turns out that all the eigenvalues of 8 are real for all positive values of δ.

In figure 1 the eigenvalues of the re-written Hamiltonian 8 as

$$H = \hat{p}^2 - (ix)^N \tag{9}$$

are plotted against the continuous real parameter N [6].

The important properties illustrated in figure 1 are that:

- For $N \geq 2$ all the eigenvalues are real and positive
- For $1 < N < 2$ there are both real eigenvalues and infinitely many complex-conjugate eigenvalue pairs
- For $N \leq 1$ the eigenvalues are all complex

The formal proof of spectral reality for this particular class of Hamiltonians can be found in [35].

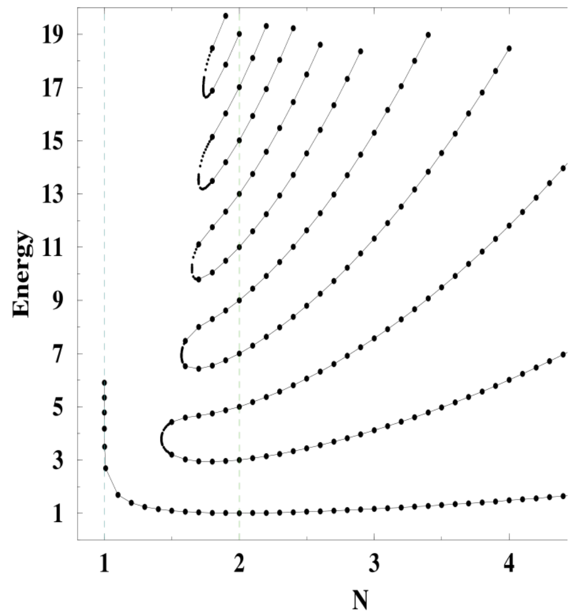

Figure 1: The eigenvalue spectrum associated with the Hamiltonian 9 as presented in [6], where the Energy eigenstates are plotted against $N \in \mathbb{R}$

2.2 Inner-Product and the $\mathcal{C}$ Operator

The useful mathematical implication of (projective) unitary operators is that they can preserve the probabilities in QM as the system evolves, where they map inner products inside of a Hilbert space $\mathcal{H} \to \mathcal{H}$. Thus, one concerning aspect of a non-Hermitian Hamiltonian is the violation of unitary evolution of the system, as attempting to create a consistent quantum theory for such Hamiltonians results in a Hilbert space equipped with an indefinite metric.

In [32] the eigenvalue problem associated with the Hamiltonian 8 proved to be a Sturm-Liouville differential equation type:

$$-\frac{d^2\phi_n(x)}{dx^2} + x^2(ix)^\delta \phi_n(x) = E_n\phi_n(x) \tag{10}$$

This differential equation must be defined along the complex x-plane, on an infinite contour denoted as C but for $0 \leq \delta < 2$ the contour can be defined on the real axis. Furthermore as in subsection 2.1,$\mathcal{PT}$ has the same eigenstates λ_n as H, ie. $\mathcal{PT}\phi_n = \lambda_n\phi_n$ and $\lambda_n = e^{ia_n}, a \in (R)$ is a pure phase. Also, completeness of eigenstates holds (as proved numerically in [35]):

$$\sum_n (-1)^n \phi_n(x)\phi_n(y) = \delta(x-y) \tag{11}$$

where $x, y \in \mathbb{R}$.

The natural guess for an inner product is:

$$\langle f|g\rangle_{\mathcal{PT}} = \int_C dx[\mathcal{PT}f(x)]g(x) \tag{12}$$

where C is the countour of integration in the x-plane and $f(x)$ and $g(x)$ are some arbitrary functions. The useful property of this inner product is that it is phase invariant and the probabilities of the quantum mechanical states are preserved. By normalizing the eigenfunctions to unity, ie. $(\phi_n, \phi_m) = 1$, it becomes apparent that the inner product 12 leads to an indefinite metric: $(\phi_n, \phi_m) = 1$. As the space and time-reflection invariant $2n \times 2n$ Hamiltonian possesses an $SU(n, n)$ symmetry, the state space of this group is one with half positive and half negative norms. This can also be concluded from the properties of the symmetry group, as it is not simply-connected.

The problem was resolved by finding a new symmetry that Hamiltonians with exact $\mathcal{PT}$ symmetry embody, described by the linear operator $\mathcal{C}$ [32], that represents the sign of the norm 12. It commutes with both the Hamiltonian and the $\mathcal{PT}$ operator and $\mathcal{C}^2 = 1$ so it has eigenvalues ± 1. The construction of $\mathcal{C}$ can be done by summing the energy eigenstates ϕ_n of the Hamiltonian:

$$\mathcal{C}(x, y) = \sum_n \phi_n(x)\phi_n(y) \tag{13}$$

Thus using this hidden symmetry, the new positive definite inner product is:

$$\langle f|g\rangle_{\mathcal{CPT}} = \int_C dx[\mathcal{CPT}f(x)]g(x) \tag{14}$$

The $\mathcal{CPT}$ inner product is also phase invariant and is positive definite, as the important property that $\mathcal{C}$ utilizes is the "sign-symmetry" of the $(+, -)$ indefinite metric of the $\mathcal{PT}$ norm, where in the negative-norm states it contributes a minus sign. The completeness condition for this norm still holds:

$$\sum_n \phi_n(x)[\mathcal{CPT}\phi_n(y)] = \delta(x - y) \tag{15}$$

An important observation for this particular inner product 14 is that it is path-independent, provided that the contour of integration C does not pass through any poles (application of Cauchy's residue theorem) or as stated in [32] C must lie inside the asymptotic wedges (described in [2]) that result from the boundary conditions of the differential equation 10, ie. $\phi(x) \to 0$ as $|x| \to \infty$.

Hence, an observable in a QM theory that has exact $\mathcal{PT}$-Symmetry can be expressed with a $\mathcal{CPT}$ invariant operator $\hat{O} \in \mathcal{H}$ (ie. $[\hat{O}, \mathcal{CPT}] = 0$).

2.2.1 Example of a $2x2$ $\mathcal{PT}$-Symmetric Hamiltonian

An example of a $2x2$ Hamiltonian, that is $\mathcal{PT}$-Symmetric and p-pseudo-Hermitian [32][36] is:

$$H = \begin{pmatrix} re^{i\theta} & s \\ t & re^{-i\theta} \end{pmatrix} \tag{16}$$

where $r, t, s, \theta \in \mathbb{R}$ and with the chosen parity operator:

$$\mathcal{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{17}$$

and time reversal operator $\mathcal{T}$ that acts as the complex conjugate, ie. $\mathcal{T}x = (x)^*$.

The Hamiltonian is of the general form that satisfies $H = \mathcal{P}^{-1}H^{\dagger}\mathcal{P}$ (pseudo-Hermicity condition), thus admits unbroken $\mathcal{PT}$ symmetry and has real eigenvalues $\epsilon_{\pm} = rcos\theta \pm \sqrt{st - r^2sin^2\theta}$ provided that $st > r^2sin^2\theta$. The eigenstates (of both H and $\mathcal{PT}$) are:

$$\epsilon_+ = \frac{1}{\sqrt{2cosa}} \begin{pmatrix} e^{i\frac{a}{2}} \\ e^{-i\frac{a}{2}} \end{pmatrix} \tag{18}$$

$$\epsilon_- = \frac{1}{\sqrt{2cosa}} \begin{pmatrix} e^{-i\frac{a}{2}} \\ -e^{i\frac{a}{2}} \end{pmatrix} \tag{19}$$

where $a = r/\sqrt{st}$.

Then, using equation 13, the $\mathcal{C}$ operator is defined as:

$$\mathcal{C} = \frac{1}{cosa} \begin{pmatrix} i\ sina & 1 \\ 1 & -i\ sina \end{pmatrix} \tag{20}$$

Using the $\mathcal{CPT}$ inner product 14, it is straightforward to show that the eigenstates are orthogonal:

$$\langle \epsilon_+ | \epsilon_- \rangle_{\mathcal{CPT}} = 0 \tag{21}$$

and also the normalization condition is satisfied:

$$\langle \epsilon_\pm | \epsilon_\pm \rangle_{\mathcal{CPT}} = 1 \tag{22}$$

The eigenvalues of $\mathcal{C}$ have exactly the same sign as the $\mathcal{PT}$ norm [32] and thus they result in a positive definite inner product that is admitted by the 2-dimentional Hilbert space.

To demonstrate this claim, let $\psi = \begin{pmatrix} a \\ b \end{pmatrix}$ with $a, b \in \mathbb{C}$ then the explicit operations are

$$\mathcal{T}\psi = \begin{pmatrix} a^* \\ b^* \end{pmatrix} \tag{23}$$

$$\mathcal{PT}\psi = \begin{pmatrix} b^* \\ a^* \end{pmatrix} \tag{24}$$

$$\mathcal{CPT}\psi = \frac{1}{\cos(a)} \begin{pmatrix} a^* + ib^* \sin(a) \\ b^* - ia^* \sin(a) \end{pmatrix} \tag{25}$$

and so writing a and b in their general forms, $a = x + iy$ and $b = v + iu$, the inner product is

$$\langle \psi | \psi \rangle_{CPT} = \frac{1}{\cos(a)}(x^2 + u^2 + 2xu \sin(a) + y^2 + v^2 - 2yv \sin(a)) \tag{26}$$

which is always positive and is equal to zero if $x = y = v = u = 0$.

This particular Hamiltonian in the limit $\theta \to 0$ becomes Hermitian and the $\mathcal{C}$ operator reduces to $\mathcal{P}$ and hence the Dirac Hermicity condition is recovered. The use of $\mathcal{C}$ is only relevant when standard QM is extended to the complex domain and the real Hamiltonian is deformed into a complex one [37].

2.2.2 A Note on the No-Signaling Principle and Bell Inequalities

Besides having real spectra and unitary evolution, another requirement for a physical quantum theory is to satisfy the no-signalling priciple [38]. It is shown in [36] that a $\mathcal{PT}$ symmetric system does not violate the no-signalling principle when using the $\mathcal{CPT}$ scheme, despite other papers suggesting so [39, 40].

In terms of the conditional probabilities, the no-signaling principle is [41]:

$$\sum_a P(a, b|A, B) = P(b|B) \tag{27}$$

where the set-up of the thought experiment goes as follows [36]: A and B exist in separate points in space and share an non-separable entangled quantum state. Then, if A knows the outcome of a measurement, then B cannot know the outcome of A and is limited to their measurement, using their particular quantum state.

As an example, take the 2×2 Hamiltonian 16 considered in the previous subsection. Following [36], the orthogonality relation 21 can be expressed in terms of the state $|\Phi_i\rangle$ where:

$$|\Phi_i\rangle = \mathcal{C}^\dagger \mathcal{P} \hat{\epsilon}_i \tag{28}$$

and thus the orthogonality relation becomes:

$$\langle \Phi_i^\pm | \epsilon^\pm \rangle_{\mathcal{CPT}_j} = \delta_{ij} \tag{29}$$

The maximally entangled wavefunction $|\epsilon_0\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ that A and B share in the time-evolved state gives two possible outcomes:

$$\epsilon_f^+ = \frac{1}{\sqrt{2}\cos(a)} \begin{pmatrix} \sin(a) \\ -i \\ -i \\ -\sin(a) \end{pmatrix} \tag{30}$$

$$\epsilon_f^- = \frac{1}{\sqrt{2}\cos(a)} \begin{pmatrix} -i \\ \sin(a) \\ -\sin(a) \\ -i \end{pmatrix} \tag{31}$$

$$\tag{32}$$

where in terms of the time-evolution operator for time $U(\tau) = \exp(-iH\tau)$, the two wavefunctions are $\epsilon_f^\pm = (U(\tau)A_\pm \otimes I)\epsilon_0$ with A_{pm} being the operators I and σ_x that A uses to send signals via the entangled wave function.

Then, the joint probability, using equation 27, that describes B's outcome depending on A measuring A_- or A_+ is (in terms of the parameter a):

$$\sum_\alpha P(a,b|A,B) = \sum_{\alpha=\pm y} \langle\epsilon_f^\pm|\, |a\rangle\langle a| \otimes |b\rangle\langle b|\, |\epsilon_f^\pm\rangle = \frac{(1 \mp \sin(a))^2}{2(1+\sin^2(a)} \tag{33}$$

where the sum runs from $\alpha = \pm y$ signifying the possible outcomes of a or b respectively.

So by looking at the joint probability result above, it is clear to see that there is a violation of the no-signaling principle as B's outcome depends on A measuring A_- or A_+. The states are not orthogonal $|\epsilon_i\rangle\langle\epsilon_j| \neq 0$ but when using the $\mathcal{CPT}$ inner product the states are indeed orthogonal (as mentioned above in equation 28) and therefore, the correct entangled states in terms of $\Phi_f^\pm = \mathcal{CP}\epsilon_f^\pm$ are:

$$|\Phi\rangle_f^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -i \\ -i \\ 0 \end{pmatrix} \tag{34}$$

$$|\Phi\rangle_f^- = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 0 \\ 0 \\ -i \end{pmatrix} \tag{35}$$

which indeed obey:

$$\langle\Phi_f^\pm|\Phi_f^\pm\rangle = 1\ ,\ \langle\Phi_f^\pm|\Phi_f^\mp\rangle = 0 \tag{36}$$

Now, replacing $\epsilon_f^\pm$ with $\Phi_f^\pm$ in equation 33 and replacing the identity with $I = \sum_{j=1}^4 |\eta_j\rangle\langle\eta_j|$, where $|\eta_j\rangle$ are the eigenvectors of the operator $|a\rangle\langle a| \otimes |b\rangle\langle b|$, the correct form of the joint probability calculation is:

$$\sum_{a} P(a,b|A,B) = \sum_{j=1}^{4} \langle \Phi_f^{\pm}|a\rangle \langle a| \otimes |b\rangle \langle b|\eta_j\rangle \langle \eta_j|\Phi_f^{\pm}\rangle = \frac{1}{2} \tag{37}$$

Hence, the no-signaling principle holds for $\mathcal{PT}$-Symmetric QM that is defined by the $\mathcal{CPT}$ inner product, as the joint probability does not include the parameter a and thus, B's measurement does not depend on A's.

Moreover, in [36] the Bell inequality is shown to be violated in conformity with ordinary QM via the CHSH(Clauser, Horne, Shimony and Holt) game framework, which is used as a simplified Bell test and further proves that the probabilities produced behave in the same manner as with regular Hermitian QM.

The validity of the $\mathcal{CPT}$ inner product 14 is therefore verified even further, by not violating the no-signaling principle and exhibiting similar behaviour as Hermitian QM in terms of winning the CHSH game. Hence, its use seems viable for the construction of a physical QM theory.

3 Pseudo-Hermicity

A Hamiltonian exhibiting exact $\mathcal{PT}$ symmetry is not sufficient for producing real spectra alone. In [3] was shown that $\mathcal{PT}$-Symmetric Hamiltonians with real spectra are a subset of pseudo-Hermitian Hamiltonians and in fact, this exact anti-linear symmetry can be proven to be equivalent to Hermicity through the pseudo-Hermitian framework. Therefore, pseudo-Hermicity can be thought to describe the most general treatment of non-Hermitian Hamiltonians with real spectra.

In his series of papers [3, 4, 5], following the work of Bender and Boetcher, Ali Mostafazadeh laid the mathematical foundations for pseudo-Hermitian QM and proved explicitly the equivalence with the $\mathcal{PT}$-Symmetric theories.

An η-pseudo-Hermitian Hamiltonian satisfies the following condition:

$$\hat{\eta}\hat{H}\hat{\eta}^{-1} = \hat{H}^{\dagger} \tag{38}$$

where $\hat{\eta}$ is a hermitian and linear automorphism [42]. The $\hat{\eta}$ operator is called the intertwining operator and exists for any operator that is diagonalizable [3]. By setting $\hat{\eta} = 1$ the usual definition of Dirac Hermicity is recovered, so pseudo-Hermicity can be thought of as a generalization of Hermicity.

It is simple to show the equivalence of the condition for exact $\mathcal{PT}$ with pseudo-Hermicity. $\mathcal{T}$ acts as complex-conjugation and $\mathcal{P}$ is a real and linear involution, such that $\mathcal{PT}^2 = 1$. Then, taking a $\mathcal{PT}$-Symmetric Hamiltonian H:

$$\hat{H} = \hat{\mathcal{PT}}\hat{H}\hat{\mathcal{PT}} = \hat{\mathcal{P}}\hat{H}^{*}\hat{\mathcal{P}} \rightarrow \hat{H} = \hat{\mathcal{P}}\hat{H}^{\dagger}\hat{\mathcal{P}} \tag{39}$$

and by multiplying by $\mathcal{P}$ from the left:

$$\hat{H}^{\dagger} = \hat{\mathcal{P}}\hat{H}\hat{\mathcal{P}}^{-1} \tag{40}$$

Hence, the Hamiltonian $\hat{H}$ is η-pseudo-Hermitian, with $\hat{\eta} = \hat{\mathcal{P}}$ (denoted as $\mathcal{P}$-pseudo-Hermitian in A.Mostafazadeh's notation [3]).

The inner product in terms of the intertwining operator $\hat{\eta}$, assuming that the Hilbert space $\mathcal{H}$ is endowed with a positive definite norm $\langle\cdot|\cdot\rangle_{(+)}$ and the Hamiltonian H is Hermitian, can be written in the form:

$$\langle\psi|\phi\rangle_{(+)} = \langle\psi|\hat{\eta_{+}}\phi\rangle \tag{41}$$

where $\psi, \phi \in \mathcal{H}$.

3.1 Equivalence with $\mathcal{PT}$-Symmetric Framework

It is easy to check that the $\mathcal{PT}$ inner product 12 is in fact the η inner product 41, for the choice $\hat{\eta} = \hat{\mathcal{P}}$:

$$\langle\hat{\psi}|\hat{\phi}\rangle_{\mathcal{PT}} = [\hat{\mathcal{PT}}\hat{\psi}]^{T}\hat{\phi} = [\hat{\mathcal{P}}\hat{\psi}^{*}]^{T}\hat{\phi} = \langle\hat{\psi}|\hat{\mathcal{P}}\hat{\phi}\rangle = \langle\hat{\psi}|\hat{\eta}\hat{\phi}\rangle = \langle\hat{\psi}|\hat{\phi}\rangle_{(+)} \tag{42}$$

For the $\mathcal{CPT}$ inner product 14, the relation $\mathcal{T}\eta_+ = \mathcal{CPT}$ is used (as proved in[43]) [1] and thus:

$$\langle\hat{\psi}|\hat{\phi}\rangle_{\mathcal{CPT}} = [\mathcal{C}\hat{\mathcal{P}}\mathcal{T}\hat{\psi}]^T\hat{\phi} = [\hat{\mathcal{T}}\hat{\eta}\hat{\psi}]^T\hat{\phi} = [\hat{\eta}\hat{\psi}^*]^T\hat{\phi} = \langle\hat{\psi}|\hat{\eta}\hat{\phi}\rangle = \langle\hat{\psi}|\hat{\phi}\rangle_{(+)} \tag{43}$$

Therefore, the inner products 12 and 14 are equivalent to the η-pseudo-Hermitian inner product 41, as claimed in Ali Mostafazadeh's work.

3.2 Calculation of $\hat{\eta}$

In this subsection, a brief overview two different ways to calculate the intertwining (or metric) operator $\hat{\eta}$ are given as presented by A.Mostafazadeh [44]. The most standard way is the so called "Spectral method" but a more numerically-driven method using perturbation theory is also useful, depending on the problem.

3.2.1 Spectral Method

The most straight-forward approach that uses the spectral representation of η as shown below:

$$\eta = \sum_{1}^{N} |\phi_n\rangle\langle\phi_n| \tag{44}$$

This can be easily done numerically by summing over the eigenvectors ϕ_n (or in the case of a continuous spectrum, calculating the corresponding integrals).

3.2.2 Perturbative Expansion

This method is an application of perturbation theory and the sketch of the steps goes as follows:

1. Decompose the Hamiltonian H and introduce an infinitesimal parameter ϵ:

$$H = H_0 + \epsilon H_1 \tag{45}$$

   where H_0 is the Hermitian part of H and H_1 is the anti-Hermitian part.

2. As η is a positive-definite operator, its logarithm can be written as the Hermitian operator $Q = -ln\eta$ such that the pseudo-Hermicity condition can be expressed as:

$$H^\dagger = e^{-Q}He^{Q} \Rightarrow \eta = e^{-Q} \tag{46}$$

[1] Note that in [43] when referring to the $\mathcal{CPT}$ inner product, the distinction between $\hat{\eta}_+$ and $\hat{\eta}$ is that when using $\hat{\eta}_+$, a positive definite metric for the inner product $\langle\cdot|\cdot\rangle_+$ is assumed but for the current discussion the subscript is omitted.

Now, using the Baker–Campbell–Hausdorff formula on the above equation:

$$e^{-Q}He^{Q} = H + \sum_{l=1}^{\infty} \frac{1}{l!}[H,Q]_l \tag{47}$$

where $[H,Q]_l = [[...[[H,Q],\underbrace{Q,Q,....],Q}_{1,2,3,...,l}]$.

Hence, the pseudo-Hermicity condition 46 now reads:

$$H^\dagger = H + \sum_{l=1}^{\infty} \frac{1}{l!}[H,Q]_l \tag{48}$$

3. Then, using the infinitesimal parameter ϵ to expand Q:

$$Q = \sum_{j=1}^{\infty} Q_j \epsilon^j \tag{49}$$

4. Now, combining equations 45 with 49 and substituting them into equation 48, leads to a set of equations for the Hermitian operator Q_j that are of the form:

$$[H_0, Q_j] = R_j \tag{50}$$

where $j \in \mathbb{Z}^+$ and R_j (with $k < j$) is:

$$R_j = \begin{cases} -2H_1 & \text{for } j = 1. \\ \sum_{k=2}^{j}[\sum_{m=1}^{k}\sum_{n=1}^{m} \frac{(-1)^n n^k m!}{k! 2^{m-1} n!(m-n)!} & \\ \sum_{\substack{s_1,..,s_k \in \mathbb{Z}^+ \\ s_1+...+s_k=j}} [[...[H_0, Q_{s_1}], Q_{s_2}], ..., Q_{s_k}]] & \text{for } j \geq 2\ . \end{cases} \tag{51}$$

Up to and including Q_3, the explicit terms of the sum are:

$$[H_0, Q_1] = -2H_1 \tag{52}$$

$$[H_0, Q_2] = 0 \tag{53}$$

$$[H_0, Q_3] = -\frac{1}{6}[H_1, Q_1]_2 \tag{54}$$

5. Finally, the solutions in of the above equations for Q_j can be obtained.

To be concrete, consider the following example of a $\mathcal{PT}$-Symmetric cubic anharmonic oscillator [44]:

$$H = \frac{1}{2m}p^2 + \frac{1}{2}\mu^2 x^2 + i\epsilon x^3 \tag{55}$$

As defined in [45], the Q_j operators for $Q_{2i} = 0$ with $i \in \mathbb{Z}^+$ can be written as:

$$Q_{2i+1} = \sum_{j,k=0}^{i+1} c_{ijk} \{x^{2j}, p^{2k+1}\} \tag{56}$$

where the curly brackets denote the anti-commutator ($\{a, b\} = ab + ba$) and $c_{ijk} \in \mathbb{R}$ are constants. So using the above equation in 50 and following the given steps, the result for up to order $O(\epsilon^2)$ is:

$$H = \frac{1}{2m}p^2 + \frac{1}{2}\mu^2 x^2 + \frac{3\epsilon^2}{2\mu^4}\left(\frac{2}{m}x^2 p^2 + \mu^2 x^4\right) + O(\epsilon^3) \tag{57}$$

3.3 Pseudo-Unitarity

In contrast with Hermitian QM, unitary time-evolution in the pseudo-Hermitian framework is a condition purely based on the intertwining operator $\hat{\eta}$, where an operator $\hat{U} : \mathcal{H} \to \mathcal{H}$ is η-pseudo-unitary if the following equation holds:

$$\hat{U}^\dagger \hat{\eta} \hat{U} = \hat{\eta} \tag{58}$$

By considering the usual Schroedinger picture operator $\hat{U} = e^{-i\hat{H}t}$ it is clear to see that although the "ket" vector is indeed η-pseudo-unitary for an η-pseudo-Hermitian Hamiltonian, the "bra" is not, as it evolves according to $\hat{H}^\dagger$ and the Hamiltonian is not Hermitian.

This is solved by the inner product 41, as it is invariant under time translations and the probability density is conserved:

$$\langle \psi(\mathbf{x}, t) | \psi(\mathbf{x}, t) \rangle_{(+)} = \langle \psi(\mathbf{x}, t) | \hat{\eta_+} \psi(\mathbf{x}, t) \rangle = \int dx \mathcal{P}(\mathbf{x}, t) \tag{59}$$

By re-examining this result, E. Sablevice and P.Millington [33] introduced the 'dual' wavefunction $\tilde{\psi}^*$ that makes the representation of time-translations for the "bra" feasible. Hence, the probability density can be described by ψ and $\tilde{\psi}^*$, and transforms in the same representation on the Hilbert space.

The representation on a dual Hilbert space is described by the operator $\hat{U}^* : \mathcal{H}^* \to \mathcal{H}^*$ acting on the states $\langle \tilde{\phi} | \in \mathcal{H}^*$ as:

$$\langle \tilde{\phi} | \to \hat{U}^*(t) \langle \tilde{\phi} | := \langle \tilde{\phi} | \hat{U}^{-1}(t) \tag{60}$$

and the 'dual' states transform as:

$$\langle \tilde{\phi} | : \mathcal{H} \to \mathbb{C} \tag{61}$$

$$| \psi \rangle \to \langle \tilde{\phi} | \psi \rangle \tag{62}$$

in the dual representation. Furthermore, the dual "bra" states are related to "bra" states by $\langle \tilde{\phi} | = \langle \phi | \hat{\eta}$.

Thus, now a construction of consistent time evolution can be carried out, that is governed by only the Hamiltonian H. The dual wavefunction $\tilde{\psi}^*$: $\mathcal{F}(\mathcal{H}^*) \to \mathcal{F}(\mathcal{H}^*)$ is then:

$$\tilde{\psi}^*(\mathbf{x}, t) := \langle \tilde{\psi}^*(t)|\mathbf{x}\rangle = \langle \psi(0)|\hat{\eta} e^{i\hat{H}t}|\mathbf{x}\rangle \tag{63}$$

With this similar treatment, the pseudo-Hermitian time-evolution can be extended to fields (see next chapter) and describe the cannonical operators for a non-Hermitian Quantum Field Theory. Of course the above arguments are not relativistic and hence they need to be applied for the symmetry group of QFT, the Poincarè group.

4 Non-Hermitian Field Theory

In this section, the treatment of a field theory equipped with a non-Hermitian mass terms is discussed. More specifically, the redefinition of the field $\hat{\phi}$, as used in [24, 46] is shown and the resulting Euler-Lagrange solution consequences of the non-Hermitian term are discussed. Next, an alternative construction is described, by constructing a new 'dual' field $\hat{\hat{\phi}}^\dagger(x)$ that transforms under the full proper Poincarè group and the field for the same Scalar Field Lagrangian (as previous) is constructed.

4.1 Non-Hermitian Scalar Field Theory

The archetypal type of a non-Hermitian but $\mathcal{PT}$-Symmetric Lagrangians studied in [46, 33, 24] are of the form:

$$\mathcal{L}(x) = \partial^\mu \phi^\dagger(x) \partial_\mu \phi(x) - \phi^\dagger(x) M^2 \phi(x) \tag{64}$$

where the non-Hermicity stems from the mass matrix $M^2 \neq M^{2\dagger}$.

The mass term is of the form:

$$M^2 = \begin{pmatrix} m_1^2 & \mu^2 \\ -\mu^2 & m_2^2 \end{pmatrix} \tag{65}$$

where taking the limit $\lim_{\mu^2 \to 0}$ recovers Hermicity.

4.1.1 $\mathcal{P}$ and $\mathcal{T}$ Transformations

Let the scalar field $\phi(x)$ have two complex components, one be a scalar and the other a pseudo-scalar:

$$\phi(t, \mathbf{x}) = \begin{pmatrix} \phi_1(t, \mathbf{x}) \\ \phi_2(t, \mathbf{x}) \end{pmatrix} \tag{66}$$

The parity transformation acts on the field $\phi(x)$ as:

$$\mathcal{P}: \qquad \phi(t, \mathbf{x}) \rightarrow \phi'(t, -\mathbf{x}) = e^{i\gamma_1} \phi(x) \tag{67}$$

$$\begin{pmatrix} \phi_1(t, \mathbf{x}) \\ \phi_2(t, \mathbf{x}) \end{pmatrix} \rightarrow \begin{pmatrix} \phi_1'(t, -\mathbf{x}) \\ \phi_2'(t, -\mathbf{x}) \end{pmatrix} = \begin{pmatrix} e^{ia_1} & 0 \\ 0 & e^{ia_2} \end{pmatrix} \begin{pmatrix} \phi_1(t, \mathbf{x}) \\ -\phi_2(t, \mathbf{x}) \end{pmatrix} \tag{68}$$

where $\gamma_1, a_i \in \mathbb{R}$.

The time-reversal transformation acts on the field as:

$$\mathcal{T}: \qquad \phi(t, \mathbf{x}) \rightarrow \phi'(-t, \mathbf{x}) = e^{i\gamma_2} \phi^*(x) \tag{69}$$

$$\begin{pmatrix} \phi_1(t, \mathbf{x}) \\ \phi_2(t, \mathbf{x}) \end{pmatrix} \rightarrow \begin{pmatrix} \phi_1'(-t, \mathbf{x}) \\ \phi_2'(-t, \mathbf{x}) \end{pmatrix} = \begin{pmatrix} e^{ib_1} & 0 \\ 0 & e^{ib_2} \end{pmatrix} \begin{pmatrix} \phi_1^*(t, \mathbf{x}) \\ \phi_2^*(t, \mathbf{x}) \end{pmatrix} \tag{70}$$

where $\gamma_2, b_i \in \mathbb{R}$.

One consequence of non-Hermitian potentials is that they display a certain behaviour when sources and sinks are present. More specifically, under the action of the $\mathcal{T}$ operator the sinks become sources (due to the change in time flow) and thus, under $\mathcal{PT}$ action, the system is symmetric up to a phase [24]:

$$\mathcal{PT}: \begin{pmatrix} \phi_1(t,\mathbf{x}) \\ \phi_2(t,\mathbf{x}) \end{pmatrix} \to \begin{pmatrix} \phi_1'(t,\mathbf{x}) \\ \phi_2'(t,\mathbf{x}) \end{pmatrix} = e^{i\gamma} \begin{pmatrix} \phi_1^*(t,\mathbf{x}) \\ -\phi_2^*(t,\mathbf{x}) \end{pmatrix} \tag{71}$$

where $\gamma \in \mathbb{R}$.

The eigenvalues $M_\pm$ of M^2 are:

$$M_\pm = \frac{1}{2}\left(m_1^2 + m_2^2 \pm \sqrt{(m_1^2 - m_2^2)^2 - 4\mu^4}\right) \;, M_\pm \in \mathbb{R} \text{ for } \frac{|m_1^2 - m_2^2}{2} \leq |\mu^2| \tag{72}$$

where the unbroken region of $\mathcal{PT}$-Symmetry corresponds to eigenvalues being real.

4.1.2 Euler-Lagrange Equations

The equations of motion (eom) for complex-component Hermitian Lagrangian are straight forward to calculate:

$$\frac{\delta S}{\delta \phi_i} = \frac{\partial S}{\partial \phi_i} - \partial_a \left(\frac{\partial S}{\partial(\partial_a \phi_i)}\right) = 0 \tag{73}$$

$$\frac{\delta S}{\delta \phi_i^*} = \frac{\partial S}{\partial \phi_i^*} - \partial_a \left(\frac{\partial S}{\partial(\partial_a \phi_i^*)}\right) = 0 \tag{74}$$

where ϕ_i is an n-component field with $i = \{1, 2, .., n\}$ and the action is defined as usual, ie. $S = \int d^4x\mathcal{L}$.

As the Lagrangian is Hermitian, then the following relation holds:

$$\mathcal{L} = \mathcal{L}^\dagger \Rightarrow \left(\frac{\delta S}{\delta \phi_i}\right)^* = \frac{\delta S^*}{\delta \phi_i^*} = \frac{\delta S}{\delta \phi_i^*} \tag{75}$$

thus, this is the only contraint for the eom 73 and 74.

When a non-Hermitian Lagrangian is considered, the relation 75 above is no longer valid and the eom can no longer be non-trivially defined. Hence, as proposed first in [46], the $\mathcal{PT}$-Symmetry of the system can be exploited and re-define the Lagrangian in terms of $\mathcal{PT}$-conjugate fields in the following notation:

$$\Phi = \begin{pmatrix} \phi_1(t,\mathbf{x}) \\ \phi_2(t,\mathbf{x}) \end{pmatrix}, \Phi_{\mathcal{PT}}^\dagger \equiv [\mathcal{PT}\Phi]^T = (\phi_1^*(t,\mathbf{x}) \;,\; -\phi_2^*(t,\mathbf{x})) \tag{76}$$

so now, the Lagrangian 64 becomes:

$$\mathcal{L}(x) = \Phi_{\mathcal{PT}}^\dagger \begin{pmatrix} -\Box - m_1^2 & -\mu^2 \\ -\mu^2 & \Box + m_2^2 \end{pmatrix} \Phi \tag{77}$$

where $\Box = \partial^\mu \partial_\mu$ denotes the d'Alembertian.

The eom can now be defined by the re-written fields:

$$\frac{\delta S}{\delta \Phi} = \left(\frac{\delta S}{\delta \Phi^\dagger_{\mathcal{PT}}}\right)^\dagger_{\mathcal{PT}} = \begin{pmatrix} -\Box\phi_1^* - m_1^2\phi_1^* + \mu^2\phi_2^* \\ -\Box\phi_2^* - m^2\phi_2^* - \mu^2\phi_1^* \end{pmatrix}^T = 0 \tag{78}$$

$$\frac{\delta S}{\delta \Phi^\dagger_{\mathcal{PT}}} = \begin{pmatrix} -\Box\phi_1 - m_1^2\phi_1 - \mu^2\phi_2 \\ \Box\phi_2 + m_2^2\phi_2 - \mu^2\phi_1 \end{pmatrix} = 0 \tag{79}$$

and the other pair of eom can be obtained by conjugating the above:

$$\left(\frac{\delta S}{\delta \Phi}\right)^* = \begin{pmatrix} -\Box\phi_1 - m_1^2\phi_1 + \mu^2\phi_2 \\ -\Box\phi_2 - m^2\phi_2 - \mu^2\phi_1 \end{pmatrix}^T = 0 \tag{80}$$

$$\left(\frac{\delta S}{\delta \Phi^\dagger_{\mathcal{PT}}}\right)^* = \begin{pmatrix} -\Box\phi_1^* - m_1^2\phi_1^* - \mu^2\phi_2^* \\ \Box\phi_2^* + m_2^2\phi_2^* - \mu^2\phi_1^* \end{pmatrix} = 0 \tag{81}$$

These equations do not allow for "dynamical" solution to the Euler-Lagrange equations and thus, only one equation must be chosen to proceed, although either choice would lead to the same observables. This is an obvious result of non-Hermicity, as the eom 73,74 can have a trivial solution $\phi = \phi^* = 0$ or only one equation can be non-trivially defined.

As for the conserved currents, starting from the Hermitian limit $\mu \to 0$, the two $U(1)$ currents (one for each field) are:

$$j_a^\nu = i\left(\phi_a^* \partial^\nu \phi_a - \phi_a \partial^\nu \phi_a^*\right) \text{ with } a = 1, 2 \tag{82}$$

but in the non-Hermitian limit $\mu \neq 0$ these currents are not conserved, but their difference is conserved under the eom, ie.:

$$j^\nu = j_1^\nu - j_2^\nu \tag{83}$$

Although this "composite" current is conserved, it is not a symmetry of the Lagrangian, as the corresponding $U(1)$ transformations are:

$$\phi_1(x) \to \phi_1'(x) = e^{i\gamma_1}\phi_1(x) \tag{84}$$

$$\phi_2(x) \to \phi_2'(x) = e^{-i\gamma_2}\phi_2(x) \quad \text{ with } \gamma_i \in \mathbb{R} \tag{85}$$

and the Lagrangian is not left invariant under the above, it is left invariant only under a single $U(1)$ transformation:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \to e^{-i\gamma} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \quad \text{with} \gamma \in \mathbb{R} \tag{86}$$

These currents can be identified as the source and sink (or gain and loss) of the $\mathcal{PT}$-Symmetric potential. This points to the fact that for these theories, the variational principles must be revisited to accommodate the non-Hermicity and have interpret-able results.

4.1.3 Variational Procedure and Conserved Currents

Consider now the variation of the action:

$$\begin{aligned}
\delta S &= \int d^4x \; \delta\mathcal{L}(\Phi, \Phi^\dagger_{\mathcal{PT}}, \delta_\nu\Phi, \delta_\nu\Phi^\dagger_{\mathcal{PT}}) \\
&= \int d^4x \left(\frac{\partial\mathcal{L}}{\partial\Phi} - \partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\Phi)}\right)\;\delta\Phi + \delta\Phi^\dagger_{\mathcal{PT}}\left(\frac{\partial\mathcal{L}}{\partial\Phi^\dagger_{\mathcal{PT}}} - \partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\Phi^\dagger_{\mathcal{PT}})}\right) \\
&\quad + \;\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\Phi)}\delta\Phi \;+\; \delta\Phi^\dagger_{\mathcal{PT}}\frac{\partial\mathcal{L}}{\partial(\partial_\nu\Phi^\dagger_{\mathcal{PT}})}\right)
\end{aligned} \tag{87}$$

where the last term is the variation of the current:

$$\delta j^\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\nu\Phi)}\delta\Phi \;+\; \delta\Phi^\dagger_{\mathcal{PT}}\frac{\partial\mathcal{L}}{\partial(\partial_\nu\Phi^\dagger_{\mathcal{PT}})} \tag{88}$$

In the Hermitian case, the Euler-Lagrange equations can be exploited in order to prove current conservation (as $\delta\mathcal{L} = 0$) in the usual manner. However, for a non-Hermitian theory, only one Euler-Lagrange equation can be solved non-trivially and therefore, the conserved current in this case can be determined for a continuous transformation[2] such that the variation of the Lagrangian $\delta\mathcal{L} \neq 0$ and the explicit form is:

$$\delta\mathcal{L} = \left(\frac{\partial\mathcal{L}}{\partial\Phi} - \partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\Phi)}\right)\;\delta\Phi + \delta\Phi^\dagger_{\mathcal{PT}}\left(\frac{\partial\mathcal{L}}{\partial\Phi^\dagger_{\mathcal{PT}}} - \partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\Phi^\dagger_{\mathcal{PT}})}\right) \tag{89}$$

For instance, taking the Lagrangian 64 and choosing to proceed with the $\mathcal{PT}$-conjugate field's equation of motion:

$$\frac{\delta S}{\delta\Phi^\dagger_{\mathcal{PT}}} = \frac{\partial\mathcal{L}}{\partial\Phi^\dagger_{\mathcal{PT}}} - \partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\Phi^\dagger_{\mathcal{PT}})} = 0 \tag{90}$$

Thus, the conserved current is given by the expression 89:

$$\delta\mathcal{L} = \left(\frac{\partial\mathcal{L}}{\partial\Phi} - \partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\Phi)}\right)\;\delta\Phi \tag{91}$$

$$\rightarrow \delta\mathcal{L} = 2\mu^2(\phi_2^*\delta\phi_1 - \phi_1^*\delta\phi_2) \tag{92}$$

where this is nothing but the phase transformation 84 and 85, thus this method provides a consistent variational method for non-Hermitian theories.

[2]This statement can be thought of as being the Noether's theorem equivalent for a non-Hermitian theory, as the original statement assumes a Hermitian Lagrangian.

4.2 An Alternative Construction

In this section, an alternative approach to a "healthy" non-Hermitian QFT is derived from first principles, by extending the usual Poincarè algebra to give rise to non-Hermitian generators, as shown in [33] and the new 'dual' field is defined. The key idea of this paper is to construct this 'dual' field in order to transform in the dual representation of the proper Poincarè group. In this way, the resulting QFT has a more solid theoretical foundation and also, the inconsistencies with the variational techniques described in the previous chapter are resolved.

Lastly, an example is given by constructing a scalar field theory and the Lagrangian 64 of the previous subsection is re-written in terms of the 'dual' field.

4.2.1 Poincarè Group in Non-Hermitian QFT and its Fock Space Representations

Let an η-pseudo-Hermitian Hamiltonian operator $\hat{H} : \mathcal{F} \to \mathcal{F}$ acting on the Fock space $\mathcal{F}$. Then the time evolution, which is governed by the Schroedinger equation, is trivial for the "ket" states $|\phi\rangle \in \mathcal{F}$:

$$i\partial_t |\phi(t)\rangle = \hat{H} |\phi(t)\rangle \tag{93}$$

but for the conjugate "bra" states, the same problem as in the Quantum Mechanical case arises, ie. that $\langle\phi(t)| \in \mathcal{F}$ evolves according to $\hat{H}^\dagger$. Thus, following similar arguments as before, the 'dual' field operator that can transform in the dual representation can be determined. But the complication this time is that the symmetry group of QFT is the proper Poincarè group $ISO(1,3)^\uparrow$ and the dual field of ϕ needs to be invariant under its action.

As discussed in [33], having a theory with a non-Hermitian Hamiltonian, which is the generator of time translations, means that the generators of $ISO(1,3)^\uparrow$ will be non-Hermitian as well.

The generators of the Poincarè Lie Algebra [47] for boosts/rotations and space/time translations are $J^{\mu\nu}$ and P^μ respectively. Then, for an operator $\hat{U}$ of $ISO(1,3)^\uparrow$ that is η-pseudo-unitary, the condition $\hat{U}^\dagger\hat{\eta}\hat{U} = \hat{\eta}$ holds and by expanding:

$$\hat{U}(\epsilon, \Lambda) = \hat{\mathbb{I}} + \frac{i}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu} + i\epsilon_\mu\hat{P}^\mu, \quad \text{with } \epsilon, \Lambda \in ISO(1,3)^\uparrow \tag{94}$$

becomes apparent that $\hat{J^{\mu\nu}}$ and $\hat{P}^\mu$ are η-pseudo-Hermitian themselves:

$$\hat{J}^{\dagger\mu\nu} = \hat{\eta}\hat{J^{\mu\nu}}\hat{\eta}^{-1} \tag{95}$$

$$\hat{P}^{\dagger\mu} = \hat{\eta}\hat{P}^\mu\hat{\eta}^{-1} \tag{96}$$

Hence, the generators are non-Hermitian unless they commute with $\hat{\eta}$.

4.2.2 $ISO(1,3)^{\uparrow}$ Representations in Fock Space

The proper Poincarè group can be broken into the proper Lorentz group $SO(1,3)^{\uparrow}$ and the 4-dimensional space translations $\mathbb{R}^{1,3}$, where their operators in Fock space are denoted as $\hat{U}(\Lambda)$ and $\hat{U}(\epsilon)$ respectively:

$$\hat{U}(\epsilon,\Lambda) \in ISO(1,3)^{+} : \hat{U}(\epsilon,\Lambda) = \hat{U}(\Lambda,0)\hat{U}(\epsilon,\mathbb{I})$$

and the operators are η-pseudo-unitary.

A matrix element $\mathcal{M}$ for a pseudo-Hermitian theory is defined with the inner-product 41:

$$\mathcal{M}_{ab} = \langle a|\phi(\hat{x})|b\rangle_{+} = \langle a|\hat{\eta}\hat{\phi}|b\rangle \tag{97}$$

and then the expectation value of an operator in $\mathcal{F}$ is:

$$\Phi(x) = \langle a|\hat{\phi}|a\rangle_{+} \tag{98}$$

Under the action of $\hat{U}(\epsilon,\Lambda)$, the expectation value must transform as their classical field counterparts (the so called correspondence principle [48]) and hence, an element of $ISO(1,3)^{\uparrow}$ transforms the expectation value in three distinct representations:

1. The coordinate transformation (infinite-dimentional):

$$(\epsilon,\Lambda): \quad \mathbb{R}^{1,3} \to \mathbb{R}^{1,3} \tag{99}$$
$$x \to (\epsilon,\Lambda)[x] = \Lambda x + \epsilon \tag{100}$$

2. The Fock Space representation (infinite-dimentional) by the operator $\hat{U}(\epsilon,\Lambda)$:

$$\hat{U}(\epsilon,\Lambda): \quad \mathcal{F} \to \mathcal{F} \tag{101}$$
$$|a\rangle \to \hat{U}(\epsilon,\Lambda)\,|a\rangle \tag{102}$$

3. The finite-dimentional representation of the proper Lorentz group $SO(1,3)^{\uparrow}$ by $D(\Lambda)$, of an n-component field:

$$D(\Lambda): \quad \mathbb{C}^{n} \to \mathbb{C}^{n} \tag{103}$$
$$\hat{\phi}^{a} \to D^{a}_{b}(\Lambda)\hat{\phi}^{b}, \quad \text{with } a,b = \{1,2,...,n\} \tag{104}$$

Now, a separate treatment can be applied to each counterpart of the representation (one for space-time translations and one for $SO(1,3)^{\uparrow}$), hence deducing how the expectation value $\Phi(x) \in \mathcal{F}$ transforms and finally identifying the general form of the dual field:

4.2.3 Space-time Translations $\mathbb{R}^{1,3}$

The space-time translations for the expectation value $\Phi(x)$ under $\mathbb{R}^{1,3}$ are given by:

$$\mathbb{R}^{1,3} : \Phi(x) \to \Phi'(x') = \Phi(x) \tag{105}$$

$$\Phi'(x') = \langle a'|\hat{\eta}\hat{\phi}(x')|a'\rangle = \langle a|\hat{U}^\dagger\hat{\eta}\hat{\phi}(x)\hat{U}|a\rangle = \langle a|\hat{\eta}\hat{\phi}(x)|a\rangle \tag{106}$$

By utilizing the pseudo-unitary condition and rearranging as $\hat{U}^\dagger\hat{\eta}\hat{U} = \hat{\eta} \Rightarrow \hat{U}^\dagger\hat{\eta} = \hat{\eta}\hat{U}$, the explicit transformation of the field is:

$$\hat{U}^{-1}(\epsilon)\hat{\phi}(x)\hat{U}(\epsilon) = e^{-\epsilon^\mu\partial_\mu}\hat{\phi}(x) \tag{107}$$

where $\hat{U}(\epsilon) = e^{i\epsilon^\mu\hat{P}_\mu}$ and then expanding both sides results in the following bracket:

$$[\hat{\phi}(x), \hat{P}_\mu] = i\partial_\mu\hat{\phi}(x) \tag{108}$$

describing the relationship of the translation generators and the field $\hat{\phi}(x)$ in $\mathcal{F}$.

The complex conjugate of the above two expressions gives the transformation of $\hat{\phi}^\dagger(x)$:

$$(\hat{U}^{-1}(\epsilon))^\dagger\hat{\phi}^\dagger(x)(\hat{U}(\epsilon))^\dagger = e^{-\epsilon^\mu\partial_\mu}\hat{\phi}^\dagger(x) \tag{109}$$

$$\Rightarrow (\hat{\eta}\hat{U}^{-1}(\epsilon)\hat{\eta}^{-1})\hat{\phi}^\dagger(x)(\hat{\eta}\hat{U}(\epsilon)\hat{\eta}^{-1}) = e^{-\epsilon^\mu\partial_\mu}\hat{\phi}^\dagger(x) \tag{110}$$

and

$$[\hat{\phi}(x), \hat{\eta}\hat{P}_\mu\hat{\eta}^{-1}] = i\partial_\mu\hat{\phi}^\dagger(x) \tag{111}$$

Hence now, the dual field $\hat{\tilde{\phi}}^\dagger(x)$, that transforms in the same representation as $\hat{H}$, can be defined by considering the transformation of the conjugate field above 110 and rearranging:

$$\hat{U}^{-1}(\epsilon)(\hat{\eta}^{-1}\hat{\phi}^\dagger(x)\hat{\eta})\hat{U}^(\epsilon)) = e^{-\epsilon^\mu\partial_\mu}\hat{\eta}\hat{\phi}^\dagger(x)\hat{\eta}^{-1} \tag{112}$$

and this gives:

$$[\hat{\eta}\hat{\phi}^\dagger(x)\hat{\eta}^{-1}, \hat{P}_\mu] = i\partial_\mu(\hat{\eta}\hat{\phi}^\dagger(x)\hat{\eta}^{-1}) \tag{113}$$

Thus by inspection, the general form of the dual field can be written as:

$$\hat{\tilde{\phi}}^\dagger(x) = \hat{\pi}^{-1}\hat{\phi}^\dagger(x)\hat{\pi} \tag{114}$$

4.2.4 $SO(1.3)^{\uparrow}$ transformations

For the proper Lorentz transformations, the matrix element for an n-component field $\Phi^a(x)$ (with $a = \{1, 2, ..., n\}$) transforms as:

$$SO(1,3)^{+} : \Phi(x) \to \Phi^{a\prime}(x') = D^a_b(\Lambda)\Phi^b(x) \tag{115}$$

$$\Phi^{a\prime}(x') = \langle a|\hat{U}^{\dagger}\hat{\eta}\hat{\phi}^a(x')\hat{U}|a\rangle = D^a_b(\Lambda)\,\langle a|\hat{\eta}\hat{\phi}^b(x)|a\rangle \tag{116}$$

where $D(\Lambda)$ indicates the matrix representation of $SO(1,3)^{\uparrow}$.
In a similar fashion as before, the transformation the field $\hat{\phi}(x)$ is:

$$\hat{U}^{-1}(\Lambda)\hat{\phi}^a(x)\hat{U}(\Lambda) = e^{-\frac{1}{2}\omega_{\mu\nu}m^{\mu\nu}} D^a_b(\Lambda)\hat{\phi}^b(x) \tag{117}$$

where $m_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu$ are the generators (boosts/rotations) and they are clearly Hermitian. With $\hat{U}(\Lambda) = e^{\frac{i}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu}}$ and $D^a_b(\Lambda) = e^{\frac{i}{2}\omega_{\mu\nu}(M^{\mu\nu})^a_b}$ after expanding, the resulting bracket reads:

$$[\hat{\phi}^a(x), \hat{J}^{\mu\nu}] = ((M^{\mu\nu})^a_b + im^{\mu\nu}\delta^a_b)\hat{\phi}^b(x) \tag{118}$$

For the conjugate field $\hat{\phi}^{\dagger}(x)$, the above two expressions are:

$$(\hat{\eta}\hat{U}^{-1}(\Lambda)\hat{\eta}^{-1})\hat{\phi}^{\dagger a}(x)(\hat{\eta}\hat{U}(\Lambda)\hat{\eta}^{-1}) = e^{-\frac{1}{2}\omega_{\mu\nu}m^{\mu\nu}}\hat{\phi}^{\dagger b}(x)D^{\dagger a}_b(\Lambda) \tag{119}$$

and

$$[\hat{\phi}^{\dagger a}(x), \hat{J}^{\dagger\mu\nu}] = \hat{\phi}^{\dagger b}(x)(-(M^{\dagger\mu\nu})^a_b + im^{\mu\nu}\delta^a_b) \tag{120}$$

Hence, the dual field $\hat{\hat{\phi}}^{\dagger}(x)$ can be determined by considering the transformation 119 and using the rearranged pseudo-unitarity condition $\hat{U}^{\dagger}\hat{\eta} = \hat{\eta}\hat{U}$:

$$\hat{U}^{-1}(\Lambda)(\hat{\eta}^{-1}\hat{\phi}^{\dagger a}(x)\hat{\eta})\hat{U}(\Lambda) = (e\hat{t}a^{-1}\hat{\phi}^{\dagger b}(\Lambda x)\hat{\eta})D^{\dagger a}_b(\Lambda) \tag{121}$$

and this gives:

$$[\hat{\eta}^{-1}\hat{\phi}^{\dagger}(x)\hat{\eta}, \hat{J}^{\mu\nu}] = \hat{\eta}^{-1}\hat{\phi}^{\dagger}(x)\hat{\eta}(-M^{\dagger\mu\nu} + im^{\mu\nu}) \tag{122}$$

Now, by assuming that the generators of the proper Lorentz group $M^{\mu\nu}$ are π-pseudo-Hermitian for some Hermitian matrix $\pi : \mathbb{C}^n \to \mathbb{C}^n$, such that $M^{\dagger\mu\nu} = \pi M_{\mu\nu}\pi^{-1}$ (and indeed they are as proved in [33]), the dual field transformation can be written as:

$$\hat{U}^{-1}(\Lambda)(\hat{\eta}^{-1}\hat{\phi}^{\dagger}(x)\hat{\eta}\pi)\hat{U} = (\hat{\eta}^{-1}\hat{\phi}^{\dagger}(\Lambda^{-1}x)\hat{\eta}\pi)D^{-1}(\Lambda) \tag{123}$$

which leads to:

$$[\hat{\eta}^{-1}\hat{\phi}^{\dagger}(x)\hat{\eta}\pi, \hat{J}^{\mu\nu}] = \hat{\eta}^{-1}\hat{\phi}^{\dagger}(x)\hat{\eta}\pi(-M^{\mu\nu} + im^{\mu\nu}) \tag{124}$$

Hence finally, the general form of the dual field can be defined as:

$$\hat{\tilde{\boldsymbol{\phi}}}^{\dagger}(\boldsymbol{x}) := \hat{\boldsymbol{\eta}}^{-1}\hat{\boldsymbol{\phi}}^{\dagger}(\boldsymbol{x})\hat{\boldsymbol{\eta}}\boldsymbol{\pi} \tag{125}$$

which is the main result of [33]. This expression transforms in the dual representation of $SO(1,3)^{\uparrow}$ and $\mathbb{R}^{1,3}$ and thus, is the dual of the full proper Poincarè group $ISO(1,3)^{\uparrow}$.

With this definition, the mass terms of a pseudo-Hermitian QFT, constructed by the bi-linear $\hat{\tilde{\phi}}^{\dagger}(x)\hat{\phi}(x)$ are Poincarè invariant, as well as pseudo-Hermitian.

4.2.5 Pseudo-Hermitian Representations of $SO(1,3)^{\uparrow}$

Now, consider the n-dimensional matrix representation of the proper Lorentz group:

$$D(\Lambda),\ \Lambda \in SO(1,3)^{\uparrow}: \quad \mathbb{C}^n \to \mathbb{C}^n \tag{126}$$

$$\hat{\phi}^a \to D^a_b(\Lambda)\hat{\phi}^b, \quad \text{with } a,b = \{1,2,...,n\} \tag{127}$$

where as already discussed, the Fock Space generators $J^{\mu\nu}$ and the generators of finite-dimensional representations $M^{\mu\nu}$ are non-Hermitian but π-pseudo-Hermitian for some Hermitian $n \times n$ matrix π (ie. $M^{\dagger\mu\nu} = \pi M^{\mu\nu}\pi^{-1}$).

The (complexified) Lie Algebra of of the Lorentz group $\mathfrak{so}(1,3)_{\mathbb{C}}$ can be decomposed into the direct sum of two complex Special Linear group Lie Algebras:

$$\mathfrak{so}(1,3)_{\mathbb{C}} \simeq \mathfrak{sl}(2,\mathbb{C}) \otimes \mathfrak{sl}(2,\mathbb{C}) \tag{128}$$

Hence, all finite-dimensional matrix representations of $SO(1,3)^{\uparrow}$ can be obtained by using the finite-dimensional representations of $SL(2,\mathbb{C})$ [49, 50].

The complex Special Linear group consists of 2×2 matrices M with complex entries and has the only constraint of having a unit determinant:

$$SL(2,\mathbb{C}) = \{M \in GL(2,\mathbb{C}) \mid det(M) = 1\} \tag{129}$$

and the corresponding Lie Algebra is described by traceless complex 2×2 matrices X:

$$\mathfrak{sl}(2,\mathbb{C}) = \{X \in \mathfrak{gl}(2,\mathbb{C}) \mid Tr(X) = 0\} \tag{130}$$

The generators of this algebra are the denoted by $J_a = \frac{\sigma_a}{2}$ where σ_a are the Pauli matrices and as per usual, they obey the following commutation relations [50]:

$$[J_0, J_{\pm}] = \pm J_{\pm} \tag{131}$$

$$[J_+, J_-] = -2J_0 \tag{132}$$

where the subscripts $\pm$ and 0 are common notation that stem from their "ladder operator"-like structure and J_+/J_- are the raising and lower operators respectively.

Then, considering a representation $\rho_i(J_a)$ on some vector space $\mathcal{V}_i$, a π-pseudo-Hermitian representation can be defined:

$$\rho_i(J_a)^\dagger = \pi\rho_i(J_a)\pi^{-1} \tag{133}$$

which is implied by the pseudo-Hermicity of the operators J_a and their commutation relations above, as the bracket:

$$[\rho_i(J_a)^\dagger, \rho_i(J_b)^\dagger] = \rho_i([J_a, J_b]) \tag{134}$$

preserves the composition rule (by definition of the representation, ie. homomorphism).

Now, in terms of $\mathfrak{sl}(2,\mathbb{C})$ representations, the general form of irreducible representations of $\mathfrak{so}(1,3)_{\mathbb{C}}$ can be defined, for rotation generators:

$$\kappa_{jk}(R_a) = -i[\rho_j(J_a)\otimes\mathbb{I}_{2k+1} \ + \ \mathbb{I}_{2j+1}\otimes\rho_k(J_a)] \tag{135}$$

and for boost generators:

$$\kappa_{jk}(B_a) = -\rho_j(J_a)\otimes\mathbb{I}_{2k+1} \ + \ \mathbb{I}_{2j+1}\otimes\rho_k(J_a) \tag{136}$$

where they can be combined and written as a general element: $M = a^a R_a + \beta^a B_a \ \in \mathfrak{so}(1,3)$, with $a^a, b^a \in \mathbb{R}$ such that:

$$\kappa_{jk}(M) = a^a\kappa_{jk}(R_a) + \beta^a\kappa_{jk}(B_a) \tag{137}$$

Finally, the representation of the proper Lorentz Lie Algebra can be written [33]:

$$D_{jk} = e^{\kappa_{jk}(M)} = e^{a^a\kappa_{jk}(R_a)+\beta^a\kappa_{jk}(B_a)} \tag{138}$$

4.2.6 An Example of a $\mathcal{PT}$-Symmetric Scalar Field

In section 4.1, the problems with having a non-Hermitian Lagrangian where discussed, namely the problem with the dynamical solution of the Euler-Lagrange equations, where the Lagrangian needed to be redefined by the $\mathcal{PT}$-conjugate field 110 but remained in the same representation and thus, the problem of unitary time evolution arises. Now that the dual field $\hat{\tilde{\phi}}^\dagger(x)$ has been obtained, the "correct" Lagrangian as used in [33] can be written as:

$$\hat{\mathcal{L}}(x) = \partial^\mu\hat{\tilde{\phi}}^\dagger(x)\partial_\mu\phi(\hat{x}) - \hat{\tilde{\phi}}^\dagger(x)M^2\phi(\hat{x}) \tag{139}$$

instead of the "naive" Lagrangian 64 previously considered [24].

The scalar field transforms in the trivial representation (as a pure number) and using equation 138, the representation matrix $D(\Lambda)$ is:

$$D_{00}(\Lambda) = e^0\mathbb{I} = \mathbb{I}_{n\times n}, \text{ for all } \Lambda \in SO(1,3)^\uparrow \tag{140}$$

Therefore, $D_{00}^\dagger = \pi D_{00}\pi^{-1} = \pi\pi^{-1} = 1,\ \pi \in \mathbb{R}$ is π-pseudo-unitary and the dual field operator form is, according to equation 125:

$$\hat{\tilde{\phi}}^\dagger(x) = \hat{\eta}^{-1}\hat{\phi}^\dagger(x)\hat{\eta}\Pi \tag{141}$$

with:

$$\Pi = \begin{pmatrix} \pi_1 & 0 \\ 0 & \pi_2 \end{pmatrix},\ \ \pi_1, \pi_2 \in \mathbb{R} \tag{142}$$

as the field $\hat{\phi}(x)$ has 2-components.

In contrast with its previous treatment, the eom now can be defined directly:

$$\Box\hat{\tilde{\phi}}^\dagger(x) + \hat{\tilde{\phi}}^\dagger(x)M^2 = 0 \tag{143}$$

with solution (in momentum space):

$$\hat{\tilde{\phi}}^\dagger(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}}[\hat{a}^\dagger(0,\boldsymbol{p})e^{ip\cdot x} \ + \ \hat{c}(0,\boldsymbol{p})e^{-ip\cdot x}] \tag{144}$$

where $E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2\mathbb{I} + M^2}$ and it is related to the field $\hat{\phi}(x)$ with the equation 125:

$$\hat{\tilde{\phi}}^\dagger(x) = \hat{\mathcal{P}}^{-1}\hat{\phi}^\dagger(x^P)\hat{\mathcal{P}}P \tag{145}$$

where P denotes the parity matrix $P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (see Appendix B: subsection 6.2 for the action of $\mathcal{P}$ and $\mathcal{T}$ on $\hat{a}$ and $\hat{c}$).

A naive but obvious choice for the intertwining operator $\hat{\eta}$ is the parity operator $\hat{\mathcal{P}}$ but the inner product $\langle\cdot|\hat{\mathcal{P}}\cdot\rangle$ is not invariant under space translations in Fock space:

$$\langle a|\hat{\mathcal{P}}|b\rangle \rightarrow \langle a|e^{-i\epsilon_a\hat{P}^{\dagger a}}\hat{\mathcal{P}}e^{i\epsilon_a\hat{P}^a}\rangle \tag{146}$$

as the operator $\hat{\mathcal{P}}$ flips the sign of the 3-momentum operator $\hat{P}^a$, ie. $\hat{\mathcal{P}}^{-1}\hat{\boldsymbol{P}}\hat{\mathcal{P}} = -\hat{\boldsymbol{P}}$.

As discussed in the leading subsections, in order for the space translation operator to remain Hermitian, it needs to commute with the intertwining operator $[\hat{\eta}, \hat{P}^a] = 0$. Hence, the right choice cannot be the parity operator$\hat{\mathcal{P}}$ as $[\hat{\mathcal{P}}, \hat{P}] \neq 0$ and another choice for $\hat{\eta}$ is needed, while also keeping in mind that the Lagrangian must remain $\hat{\eta}$-pseudo-Hermitian.

As with the $\mathcal{PT}$-Symmetric QM, the symmetry described by the $\mathcal{C}$ operator can be exploited in order to acquire an invariant inner product under space-translations and the appropriate $\hat{\eta}$ can be defined.

The construction of the matrix C, as proposed in [33], can be done by summing the eigenstates ϕ_n of the mass matrix M:

$$\mathcal{C} = \sum_n |\phi_n\rangle \langle\phi_n| P \tag{147}$$

Although this construction can be done numerically for the operator $\hat{\mathcal{C}}$, it requires the construction of a biorthonormal basis for the Fock space $\mathcal{F}$ and then the resulting sum will be infinite, as there are infinitely many states $\hat{\phi} \in \mathcal{F}$. An easier way for this $\mathcal{PT}$-Symmetric Lagrangian 139 is to identify that the $\hat{\mathcal{C}}$ operator can be taken to be the parity operator $\hat{\mathcal{P}}$ but in the basis where M^2 is non-diagonal, as the equation above 147 dictates. Assuming that the mass matrix M^2 is diagonalizable, then the Hamiltonian is diagonalizable and therefore exists a $\hat{C}$ operator such that $[\hat{H}, \hat{C}] = 0$ and $\hat{C}^2 = \mathbb{I}$ [33]:

$$\hat{H} = \int \frac{d^3p}{(2\pi)^3} \left(\hat{a}^\dagger(0,\boldsymbol{p}) E_{\boldsymbol{p}} \hat{a}(0,\boldsymbol{p}) + c^\dagger(0,\boldsymbol{p}) E_{\boldsymbol{p}} \hat{c}(0,\boldsymbol{p})\right) \Rightarrow \hat{\mathcal{C}}^{-1}\hat{H}\hat{\mathcal{C}} = \hat{H} \tag{148}$$

as $\hat{\mathcal{C}}$ commutes with $E_{\boldsymbol{p}}$ and $\hat{\mathcal{C}}^2 = \mathbb{I}$.

Using the Baker-Campbell-Hausdorf formula (in a similar fashion as in section 3.2.2), the form of the operator $\hat{\mathcal{C}}$ can be determined. First, let two operators $\hat{A}$ and $\hat{B}$:

$$\hat{A} = \int \frac{d^3p}{(2\pi)^3} \hat{a}^\dagger(0,\boldsymbol{p})\ A\ \hat{a}(0,\boldsymbol{p}) \tag{149}$$

$$\hat{B} = \int \frac{d^3p}{(2\pi)^3} \hat{a}^\dagger(0,\boldsymbol{p})\ B\ \hat{a}(0,-\boldsymbol{p}) \tag{150}$$

where matrices A and B commute ($[A, B] = 0$). Then, it can be shown that:

$$\begin{aligned} e^{ia\hat{A}} e^{b\hat{B}} \hat{a}(0,\boldsymbol{p}) e^{-ib\hat{B}} e^{-ia\hat{A}} &= \cos(bB)[\cos(aA) - i\sin(aA)]\hat{a}(0,\boldsymbol{p}) \\ &\quad - i\sin(bB)[\cos(aA) - i\sin(aA)]\hat{a}(0,-\boldsymbol{p}) \end{aligned} \tag{151}$$

$$\begin{aligned} e^{ia\hat{A}} e^{b\hat{B}} \hat{a}^\dagger(0,\boldsymbol{p}) e^{-ib\hat{B}} e^{-ia\hat{A}} &= \hat{a}^\dagger(0,\boldsymbol{p})[\cos(aA) + i\sin(aA)]\cos(bB) \\ &\quad + i\hat{a}^\dagger(0,-\boldsymbol{p})[\cos aA + i\sin(aA)]\sin(bB) \end{aligned} \tag{152}$$

and choosing the parameters for $\hat{\mathcal{C}}$ as $A = C$ (where C is just the matrix of $\hat{\mathcal{C}}$), $a = -\frac{\pi}{2}$, $B = \mathbb{I}$ and $b = \frac{\pi}{2}$, the resulting expression is:

$$\hat{\mathcal{C}}\hat{a}(0,\boldsymbol{p})\hat{\mathcal{C}}^{-1} = C\hat{a}(0,-\boldsymbol{p}) \tag{153}$$

$$\hat{\mathcal{C}}\hat{a}^\dagger(0,\boldsymbol{p})\hat{\mathcal{C}}^{-1} = \hat{a}^\dagger(0,-\boldsymbol{p})C \tag{154}$$

It is worth noting that the above action is identical to that of the parity operator $\hat{\mathcal{P}}$ but the major difference is that it commutes with the Hamiltonian.

The useful property of the $\hat{C}$ operator is, like in the case with $\hat{\mathcal{P}}$, that it flips the sign of the operator $\hat{P}^a$ and thus, the correct choice for the intertwining operator is:

$$\hat{\eta} = \hat{\mathcal{P}}\hat{C} \Rightarrow (\hat{\mathcal{P}}\hat{C}\ \hat{P}^a\ \hat{\mathcal{P}}\hat{C}) = \hat{P}^a \tag{155}$$

and consequently, the matrix Π is:

$$\Pi = PC \tag{156}$$

where the restriction implied on the given form of Π 142 is $\pi_1 = -\pi_2$. The appropriate inner product is therefore $\langle\cdot|\hat{\mathcal{P}}\hat{C}\cdot\rangle$ and is indeed invariant under spacetime translations.

Finally, the dual field is then:

$$\hat{\tilde{\phi}}^\dagger(x) = (\hat{\mathcal{P}}\hat{\mathcal{C}})^{-1}\ \hat{\phi}^\dagger(x)\ (\hat{\mathcal{P}}\hat{\mathcal{C}})PC \tag{157}$$

and the scalar field theory construction is complete in a consistent manner.

5 Conclusion

In this work, the notion of $\mathcal{PT}$-Symmetry and pseudo-Hermicity are explored, in the context of Quantum Theory. Namely, the use of complex Hamiltonians that exhibit exact (or unbroken) $\mathcal{PT}$-Symmetry are shown to produce real spectra and the new inner product 14, that uses the symmetry described by the $\mathcal{C}$ operator, is shown to produce positive probabilities. The equivalence with the more general pseudo-Hermitian framework is discussed and the equivalence with the η-inner product 41 is shown. The positive-definite $\mathcal{CPT}$ inner product shows the signs of a "healthy" QM theory, as it obeys the no-signaling principle [36], despite previous claims.

In the final chapter, the ideas developed are generalized to field theories with non-Hermitian mass terms. The well-studied route of defining a new field 110, that utilizes the $\mathcal{PT}$-Symmetry of the Lagrangian [24] is discussed and the corresponding variational method is shown. Although this method has been proved to work (even when considering fermions), a new approach is given that builds field that transforms in the 'dual' representation of the proper Poincarè group [33]. This method ensures that the time-evolution of each field depends on the Hamiltonian and not its conjugate, as is the case when considering the "naive" Lagrangian 64 of the first method. Thus, (pseudo-unitary) time evolution can be properly defined and the variational methods are more straight forward. More importantly, this method provides a solid foundation for building QFT's from first principles and can prove useful when trying to apply standard techniques to a non-Hermitian theory.

As $\mathcal{PT}$-Symmetric Hamiltonians belong to the class of pseudo-Hermitian Hamiltonians, the question of whether considering these theories instead of just generalizing the discussion into pseudo-Hermicity arises. As was discussed in section 3, the requirement for a Hamiltonian to have exact $\mathcal{PT}$-Symmetry is equivalent to the requirement of Hermicity when considering pseudo-Hermicity. Furthermore, if the Hilbert space is finite dimensional, then a Hamiltonian can have unitary time evolution, regardless of it being Hermitian or not. For an infinite dimensional vector space, a non-Hermitian Hamiltonian can be mapped onto its Hermitian counterpart [5], but locallity of the Hamiltonian operator is lost. More specifically, (as pointed out in [5]) a non-Hermitian Hamiltonian operator that is local, equipped with a (positive-definite) inner product can produce unitary evolution with respect to a different inner product, only if its Hermitian counterpart is Hermitian with respect to the initial inner product but it is generally non-local.

Although the claim that $\mathcal{PT}$-Symmetry is a more physical argument than Hermicity [6] is valid in a sense, the study of these types of Hamiltonians may be much more valuable in the context of non-local field theories, as potentially a $\mathcal{PT}$-Symmetric Hamiltonian (that is local) can be physically equivalent to a non-local Hamiltonian. Hence, the field of non-local field theories can benefit from such treatment, as they become much simpler to study.

There are still many problems to be resolved with non-Hermitian (and $\mathcal{PT}$-

Symmetric) field theories [51, 52], before reaching the point of "phenomenological" results. One major problem described in [52] is the violation of causality, when cosnidering a 2×2 model of non-Hermitian scattering. Although the calculations seemed promising, with even defining the Lehmann–Symanzik–Zimmerman (LSZ) reduction formula (in this particular example there were no problems defining such an expression, as the non-Hermitian Hamiltonian was local and thus pertubation theory could be applied as usual), the final result of the scattering amplitude violated causality as the resulting poles were in the wrong direction.

All in all, $\mathcal{PT}$-Symmetry and pseudo-Hermicity have seen great results in the recent years but the field is still in its infancy. With many papers currently being published, the problems with formulating a consistent theory could some day be solved and then a new candidate for the "final theory" could emerge.

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6 Appendices

6.1 Appendix A: Canonical Commutation Relations for $\mathcal{P}$ and $\mathcal{T}$

When $\mathcal{P}$ acts on the $[\hat{x}, \hat{p}]$, it preserves the commutation relation:

$$\mathcal{P}[\hat{x},\hat{p}]\mathcal{P}^{-1} = \mathcal{P}\hat{x}\mathcal{P}^{-1}\mathcal{P}\hat{p}\mathcal{P}^{-1} - \mathcal{P}\hat{p}\mathcal{P}^{-1}\mathcal{P}\hat{x}\mathcal{P}^{-1} = (-\hat{x})(-\hat{p}) - (-\hat{p})(-\hat{x}) = i\hbar$$

It is a linear (unitary) operator, whereas $\mathcal{T}$ must be an anti-linear operator:

$$\mathcal{T}[\hat{x},\hat{p}]\mathcal{T}^{-1} = \mathcal{T}\hat{x}\mathcal{T}^{-1}\mathcal{T}\hat{p}\mathcal{T}^{-1} - \mathcal{T}\hat{p}\mathcal{T}^{-1}\mathcal{T}\hat{x}\mathcal{T}^{-1} = (\hat{x})(-\hat{p}) - (-\hat{p})(\hat{x}) = -i\hbar$$

Thus, $\mathcal{T}$ is anti-linear: $\mathcal{T}(-i)\mathcal{T}^{-1} = (-i)^*$

6.2 Appendix B: Creation/Annihilation Operators under the action of $\mathcal{P}$ and $\mathcal{T}$

For the creation operator:

$$\hat{\mathcal{P}}\hat{c}(0,\mathbf{p})\hat{\mathcal{P}}^{-1} = \hat{c}(0,-\mathbf{p})P, \hat{\mathcal{P}}\hat{c}^\dagger(p,\mathbf{p})\hat{\mathcal{P}}^{-1} = P\hat{c}^\dagger(0,-\mathbf{p}) \tag{158}$$

$$\hat{\mathcal{T}}\hat{c}(0,\mathbf{p}\hat{\mathcal{T}}^{-1} = \hat{c}(0,-\mathbf{p}, \hat{\mathcal{T}}\hat{c}^\dagger(0,-\mathbf{p})\hat{\mathcal{T}}^{-1} = \hat{c}^\dagger \tag{159}$$

For the annihilation operator:

$$\hat{\mathcal{P}}\hat{a}(0,\mathbf{p})\hat{\mathcal{P}}^{-1} = P\hat{a}(0,-\mathbf{p}), \hat{\mathcal{P}}\hat{a}^\dagger(0,\mathbf{p})\hat{\mathcal{P}}^{-1} = \hat{a}^\dagger(0,-\mathbf{p})P \tag{160}$$

$$\hat{\mathcal{T}}\hat{a}(0,\mathbf{p})\hat{\mathcal{T}}^{-1} = \hat{a}(0,-\mathbf{p}), \hat{\mathcal{T}}\hat{a}^\dagger(0,\mathbf{p})\hat{\mathcal{T}}^{-1} = \hat{a}^\dagger(0,-\mathbf{p}) \tag{161}$$